Semantic Theory
Lecture 4: Type Theory 2

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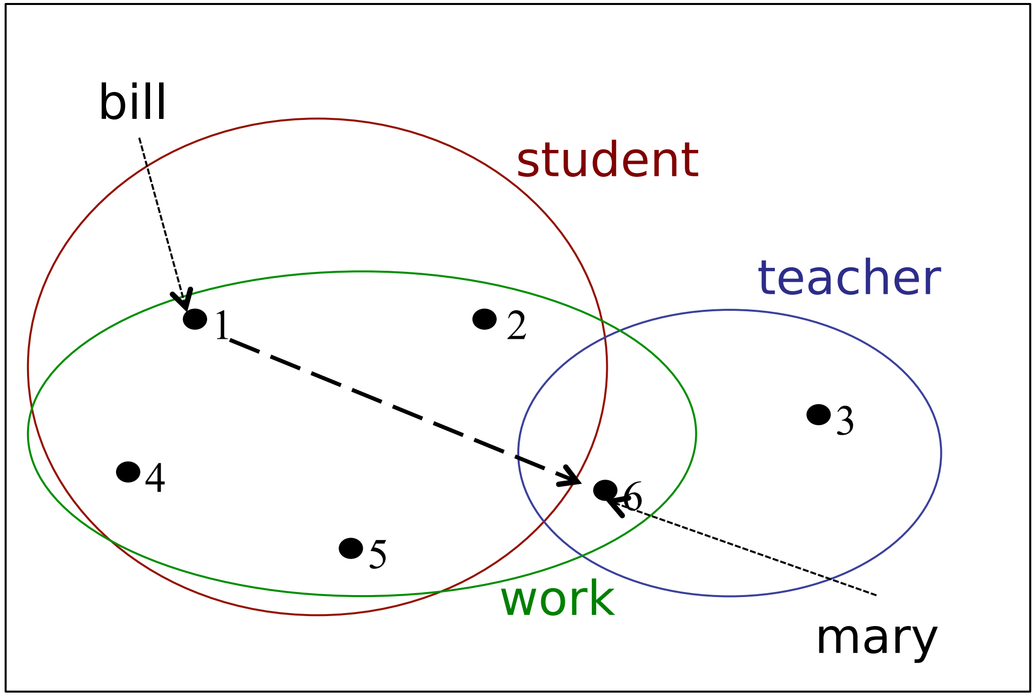
Types

- **Basic types:**
  - **e** – the type of individual terms ("**e**ntities")
  - **t** – the type of formulas ("**t**ruth-values")
- **Complex types:**
  - If σ, τ are types, then **⟨σ, τ⟩** is a type.
  - An expression of type ⟨σ, τ⟩ is a functor expression that takes a σ type expression as argument and forms a type τ expression together with it.

Type Theory – Semantics (1)

- Let U be a non-empty set of entities.
- The **domain of possible denotations** for every type τ: **D_τ** is given by:
  - $D_e = U$
  - $D_t = \{0, 1\}$
  - $D_{\langle\sigma, \tau\rangle}$ is the set of all functions from D_σ to D_τ
- Expressions of type τ denote elements of D_τ

FOL Model Structure



$$M = \langle U_M, V_M \rangle$$

$$U_M = \{ 1, 2, 3, 4, 5, 6 \}$$

$$V_M(\text{bill}) = 1$$

$$V_M(\text{mary}) = 5$$

$$V_M(\text{student}) = \{ 1, 2, 4, 5 \}$$

$$V_M(\text{teacher}) = \{ 3, 6 \}$$

$$V_M(\text{work}) = \{ 1, 2, 4, 5, 6 \}$$

$$V_M(\text{like}) = \{ \langle 1, 6 \rangle \}$$

Type-theoretic Interpretation of One-Place Predicates

- **One-place predicate** expressions have type ⟨e, t⟩.
- The set of possible denotations $D_{\langle e, t\rangle}$, is a function from entities to truth values, i.e., a member of $\{0,1\}^U$.
- In FOL, one-place predicates are represented as sets of entities.
- Functions with range {0,1} are called **characteristic functions** because they uniquely specify a subset of their domain.
- The **characteristic function** of set M in a domain U is the function $F_M: U \rightarrow \{0,1\}$ such that for all $a \in U$, $F_M(a) = 1$ iff $a \in M$.
- Type-theoretic and FOL representations of first-order predicates are equivalent.
- For practical reasons, it is often convenient to go back and forth between characteristic functions and sets.

Type-theoretic interpretation of one-place predicates: example

- For M = ⟨U, V⟩, let U consist of the persons John, Bill, Mary, Paul, and Sally. For selected types, we have the following sets of possible denotations

  $D_t = \{0,1\}$

  $D_e = U = \{j, b, m, p, s\}$

$$D_{<e,t>} = \left\{ \begin{bmatrix} j \rightarrow 1 \\ b \rightarrow 0 \\ m \rightarrow 1 \\ p \rightarrow 0 \\ s \rightarrow 1 \end{bmatrix}, \begin{bmatrix} j \rightarrow 1 \\ b \rightarrow 0 \\ m \rightarrow 1 \\ p \rightarrow 1 \\ s \rightarrow 1 \end{bmatrix}, \begin{bmatrix} j \rightarrow 0 \\ b \rightarrow 1 \\ m \rightarrow 0 \\ p \rightarrow 0 \\ s \rightarrow 1 \end{bmatrix}, \ldots \right\}$$

- Alternative set notation:

  $D_{<e,t>} = \{ \{j,m\}, \{j, m, p, s\}, \{b, s\}, \ldots\}$

A Member of $D_{<<e,t>, <e,t>>}$

$$\begin{bmatrix} \begin{bmatrix} j \to 1 \\ b \to 0 \\ m \to 1 \\ p \to 0 \\ s \to 1 \end{bmatrix} & \to & \begin{bmatrix} j \to 1 \\ b \to 0 \\ m \to 1 \\ p \to 0 \\ s \to 0 \end{bmatrix} \\ \begin{bmatrix} j \to 0 \\ b \to 1 \\ m \to 0 \\ p \to 0 \\ s \to 1 \end{bmatrix} & \to & \begin{bmatrix} j \to 0 \\ b \to 0 \\ m \to 0 \\ p \to 0 \\ s \to 1 \end{bmatrix} \\ \begin{bmatrix} j \to 1 \\ b \to 0 \\ m \to 1 \\ p \to 1 \\ s \to 1 \end{bmatrix} & \to & \begin{bmatrix} j \to 1 \\ b \to 0 \\ m \to 1 \\ p \to 0 \\ s \to 0 \end{bmatrix} \\ & \ldots & \end{bmatrix}$$

Alternative notation:

$$\begin{bmatrix} \{j,m,s\} & \to & \{j,m\} \\ \{b,s\} & \to & \{s\} \\ \{j,m,p,s\} & \to & \{j,m\} \end{bmatrix}$$

Type Theory – Vocabulary

- **Non-logical constants:** For every type τ a (possibly empty) set of non-logical constants CON_τ (pairwise disjoint)
- **Variables:** For every type τ an infinite set of variables VAR_τ (pairwise disjoint)
- **Logical symbols:** $\forall$, $\exists$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $=$
- **Brackets:** (,)

Type Theory – Semantics (2)

- **A model structure** for a type theoretic language consists of a pair **M = ⟨U, V⟩**, where
  - U is a non-empty domain of individuals
  - V is an interpretation function, which assigns to every $\alpha \in CON_\tau$ an element of D_τ.
- **Variable assignment function** g assigns to every $v \in VAR_\tau$ an element of D_τ

Interpretation Function, Examples

$V_M(\textit{john}) = j$

$V_M(\textit{mary}) = m$

$V_M(\textit{piano player})$: $\begin{bmatrix} j \rightarrow 1 \\ b \rightarrow 0 \\ m \rightarrow 1 \\ p \rightarrow 0 \\ s \rightarrow 1 \end{bmatrix}$

$V_M(\textit{semanticist})$: $\begin{bmatrix} j \rightarrow 0 \\ b \rightarrow 1 \\ m \rightarrow 0 \\ p \rightarrow 0 \\ s \rightarrow 1 \end{bmatrix}$

$V_M(\textit{skier})$: $\begin{bmatrix} j \rightarrow 1 \\ b \rightarrow 0 \\ m \rightarrow 1 \\ p \rightarrow 1 \\ s \rightarrow 1 \end{bmatrix}$

Interpretation Function, Examples

V_M(*talented*):

$$\begin{bmatrix} \begin{bmatrix} j \to 1 \\ b \to 0 \\ m \to 1 \\ p \to 0 \\ s \to 1 \end{bmatrix} & \to & \begin{bmatrix} j \to 1 \\ b \to 0 \\ m \to 1 \\ p \to 0 \\ s \to 0 \end{bmatrix} \\ \begin{bmatrix} j \to 0 \\ b \to 1 \\ m \to 0 \\ p \to 0 \\ s \to 1 \end{bmatrix} & \to & \begin{bmatrix} j \to 0 \\ b \to 0 \\ m \to 0 \\ p \to 0 \\ s \to 1 \end{bmatrix} \\ \begin{bmatrix} j \to 1 \\ b \to 0 \\ m \to 1 \\ p \to 1 \\ s \to 1 \end{bmatrix} & \to & \begin{bmatrix} j \to 1 \\ b \to 0 \\ m \to 1 \\ p \to 0 \\ s \to 0 \end{bmatrix} \\ & \ldots & \end{bmatrix}$$

Or, alternatively:

$$\begin{bmatrix} \{j,m,s\} & \to & \{j,m\} \\ \{b,s\} & \to & \{s\} \\ \{j,m,p,s\} & \to & \{j,m\} \end{bmatrix}$$

Type Theory – Syntax

- The sets of **well-formed expressions WE_τ** for every type τ are given by:

(i) $CON_\tau \subseteq WE_\tau$ and $VAR_\tau \subseteq WE_\tau$, for every type τ

(ii) If α is in $WE_{\langle\sigma, \tau\rangle}$, β in WE_σ, then $\alpha(\beta) \in WE_\tau$.

(iii) If φ, ψ are in WE_t, then $\neg\varphi$, $(\varphi\wedge\psi)$, $(\varphi\vee\psi)$, $(\varphi\rightarrow\psi)$, $(\varphi\leftrightarrow\psi)$ are in WE_t.

(iv) If φ is in WE_t and v is a variable of arbitrary type, then $\forall v\varphi$ and $\exists v\varphi$ are in WE_t.

(v) If α, β are well-formed expressions of the same type, then $\alpha = \beta \in WE_t$.

Type Theory – Interpretation

- **Interpretation with respect to** a model structure **M = ⟨U, V⟩** and a variable assignment **g**:

(i) $[\![\alpha]\!]^{M,g} = V(\alpha)$, if α is a constant

$[\![\alpha]\!]^{M,g} = g(\alpha)$, if α is a variable

(ii) $[\![\alpha(\beta)]\!]^{M,g} = [\![\alpha]\!]^{M,g}([\![\beta]\!]^{M,g})$

(iii) $[\![\neg\varphi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 0$

$[\![\varphi \wedge \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$

$[\![\varphi \vee \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 1$ or $[\![\psi]\!]^{M,g} = 1$

…

(iv) $[\![\alpha = \beta]\!]^{M,g} = 1$ iff $[\![\alpha]\!]^{M,g} = [\![\beta]\!]^{M,g}$

(v) $[\![\exists v\varphi]\!]^{M,g} = 1$ iff there is a $d \in D_\tau$ such that $[\![\varphi]\!]^{M,g[v/d]} = 1$

$[\![\forall v\varphi]\!]^{M,g} = 1$ iff for all $d \in D_\tau$: $[\![\varphi]\!]^{M,g[v/d]} = 1$

(where v is a variable of type τ)

Interpretation: Example

John is a talented piano-player

$\Rightarrow$ talented(piano-player)(john)

$[\![\text{talented(piano-player)(john)}]\!]^{M,g} =$

$[\![\text{talented(piano-player)}]\!]^{M,g}([\![\text{john}]\!]^{M,g}) =$

$[\![\text{talented}]\!]^{M,g}([\![\text{piano-player}]\!]^{M,g})([\![\text{john}]\!]^{M,g}) = V_M$

$V_M(\text{talented})(V_M(\text{piano-player}))\ (V_M(\text{john}))$

Interpretation: Example, cont‘d

$V_M(john)$

$V_M(piano\text{-}player)$

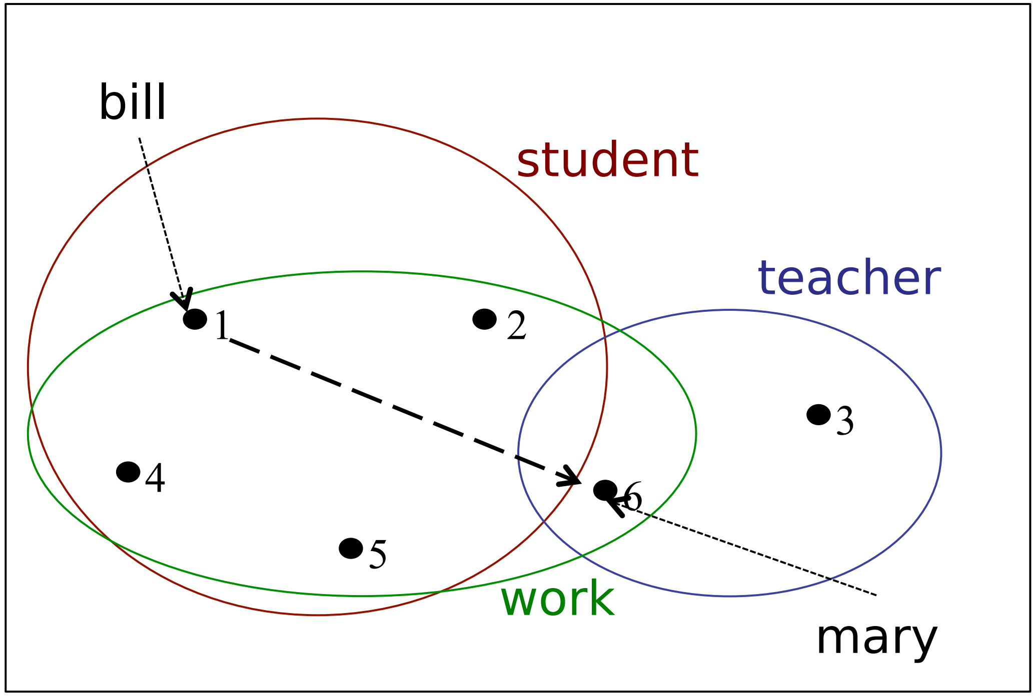
$V_M(talented)(V_M(piano\text{-}player))(j)$

$V_M(talented)(V_M(piano\text{-}player))$

$V_M(talented)$:

$$\begin{bmatrix} \begin{bmatrix} j \to 1 \\ b \to 0 \\ m \to 1 \\ p \to 0 \\ s \to 1 \end{bmatrix} & \to & \begin{bmatrix} j \to 1 \\ b \to 0 \\ m \to 1 \\ p \to 0 \\ s \to 0 \end{bmatrix} \\ \begin{bmatrix} j \to 0 \\ b \to 1 \\ m \to 0 \\ p \to 0 \\ s \to 1 \end{bmatrix} & \to & \begin{bmatrix} j \to 0 \\ b \to 0 \\ m \to 0 \\ p \to 0 \\ s \to 1 \end{bmatrix} \\ \begin{bmatrix} j \to 1 \\ b \to 0 \\ m \to 1 \\ p \to 1 \\ s \to 1 \end{bmatrix} & \to & \begin{bmatrix} j \to 1 \\ b \to 0 \\ m \to 1 \\ p \to 0 \\ s \to 0 \end{bmatrix} \\ & \ldots & \end{bmatrix}$$

Two-Place Relations



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Type-theoretic Interpretation of n-Place Relations (1)

- In type theory, functor expressions take their arguments one by one. For example, transitive verbs like *read* or *admire* expressions are analyzed as type ⟨e,⟨e, t⟩⟩ expressions:

$$\frac{\dfrac{\text{like: }\langle e,\langle e, t\rangle\rangle \quad \text{mary: } e}{\text{like(mary): }\langle e, t\rangle} \qquad \text{bill: } e}{\text{like(mary)(bill): } t}$$

- In FOL, these expressions are categorized as two-place predicates, and are assigned two-place relations $\subseteq U \times U$.
- The two variants amount to the same thing again: Type theory could easily admit and handle n-place functors, but the semantics or such a functor can be straightforwardly expressed by an expression that takes the n arguments one by one ("Currying").

Type-theoretic Interpretation of n-Place Relations (2)

- Practically, it is often convenient to write n-step sequences of function application in short as relations in FOL style. Example:

  Bill admires a talented piano player:

  Correct type-theoretic notation:

  ∃x (talented(piano-player)(x) ∧ admire(x)(bill))

  "FOL style" notation:

  ∃x (talented(piano-player)(x) ∧ admire(bill, x))

- Note that the order of arguments is different in the alternative notations: In type theory, the verb denotation is first applied to the object, because it is the innermost argument. In FOL, order of arguments usually follows the surface order of complements, so subject comes first.